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BIQUATERNION ELECTRODYNAMICS AND WEYL-CARTAN GEOMETRY OF SPACE-TIME

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The generalized Cauchy-Riemann equations (GCRE) in biquaternion algebra appear to be Lorentz-invariant. The Laplace equation is in this case replaced by a nonlinear C-eikonal equation. GCRE contain a 2-spinor and a C-gauge structures, and their integrability conditions take the form of Maxwell and Yang-Mills equations. For the value of electric charge from GCRE only the quantization rule follows, as well as the treatment of Coulomb law as a stereographic map. The equivalent geometrodynamics in a Weyl-Cartan affine space and the conjecture of a complex-quaternion structure of space-time are discussed.

1. Introduction

In the frames of the *geometrodynamic* approach all fundamental physical quantities and above all the equations of physical dynamics should be of a purely geometric nature. The twistor program, the Kaluza-Klein theories and string dynamics give representative examples of this concept, perhaps the most general ones up to now. In essence, any physical interaction may be regarded as a manifestation of geometry (by using multi-dimensional spaces, fiber bundles, etc.).

However, the diversity of admissible geometries and their invariants makes the "kinematic" part of this procedure (selection of space and geometric identification of physical quantities), as well as the dynamic one (choice of a Lagrangian) quite ambiguous. Even for the electromagnetic (EM) field one has a lot of different geometric interpretations (Weyl's conformal factor, bundle connection, the Kaluza metric field, torsion [1] or nonholonomic [2] structures of space-time (ST), etc.).

Alternatively, within the *algebrodynamic* paradigm [3] the ST is regarded as a manifold supplied with a basic algebraic structure, the structure of linear algebra in the simplest case. But it is well known that the exceptional algebras — algebras with division and positive norm — exist in the dimensions $d = 4$ (Hamilton quaternions) and $d = 8$ (Cayley octonions). So it would be natural to suppose that the ST algebra (STA) [4] should be exceptional in its internal mathematical properties. If it is the case, the group of automorphisms (Aut) of STA would generate the ST geometry, for example, by operating as an isometry group.

Moreover, the STA structure can completely determine physical dynamics as well. Indeed, if we consider the physical fields as algebra-valued functions of an algebraic variable, the *generalized Cauchy-Riemann equations* (GCRE), i.e., the differentiability conditions in the STA, become fundamental equations of field dynamics. Wonderfully, the generally accepted physical equations (in particular, the Maxwell or Yang-Mills equations) become a direct consequences of GCRE, namely their integrability conditions (see below).

From an epistemological point of view, the algebrodynamic (AD) concept returns us to the ideas of Pythagoras, Hamilton and Eddington on **a crucial role of Numbers in the structure of the Universe**. At the modern stage we deal with the primary structure of multidimensional *ST arithmetics*, completely different from the classical arithmetics of the Macroworld, or the world of reversible processes and weakly interacting objects.

A genuine ST arithmetics ought to be non-commutative and even non-associative! Indeed, these properties are just algebraic equivalents of causal and interactive structures of the physical World (ensuring the dependence of "out-state" on the order and composition of reactions). For such reasons the most suitable STA candidate is the *octonion algebra*, the unique exceptional non-associative algebra. However, the difficulties of "intercourse" with octonions are well-known (see, nevertheless, [6 7]).

Meanwhile, the non-commutativity of algebraic structures is closely connected with the non-linearity of the corresponding dynamic equations (this is the case, in particular, for the Yang-Mills fields). We will see later that the GCRE in non-commutative algebras also possess a nonlinear structure and are therefore capable to describe both quantum phenomena and physical field interactions.

In this paper we choose for a STA the *algebra of biquaternions* B , the extension of real Hamilton quaternions H to the field of complex numbers C. The H algebra is known to have Aut (H) = SO (3) and is in perfect correspondence with the structure of the 3-dimensional space. **We are unaware of a similar algebra for the case of Minkowski 4-space!** For obvious reasons one often considers the Clifford-Dirac algebra $C(1,3)$ to be the STA [4, 5]. However, a reduction from the 16-dimensional total vector space of $C(1,3)$ to a 4-dimensional physical ST is a completely "voluntaristic" procedure; even the metric signature of the basic generator space may be chosen in different ways [8].

The B-algebra, isomorphic to the Clifford algebra C(3,0) of smaller dimension $d = 8$, is preferable from this point of view. On the other hand, the B-dynamics, based on GCRE, appears to be Lorentz invariant, so **the B-algebra may be treated as a minimal STA**. This choice leads to the conjectures on a fundamental role of *null divisors* as a subspace of STA and on complex-valued structure of ST; these questions will be discussed below.

Now we are ready to present the contents of the paper. In Sec. 2 we begin with the basic definitions of the B-algebra and B-differentiability. The general problems of (bi-)quaternionic analysis are also briefly discussed. Then, in Sec. 3, after preliminary physical identifications, we demonstrate the 2-spinor structure of the basic GCRE and obtain a complexified eikonal equation for each component of the B-field. Global symmetries of the model are studied as well.

Sections 4 and 5 are devoted to B-electrodynamics as the basic case of B-differentiability. Firstly (Sec. 4) the self-duality conditions are obtained from GCRE, whence follow the Maxwell equations. Gauge invariance of a model of special type is demonstrated in Sec. 5. From the eikonal equation, a geometrical origin of the Coulomb law as a stereographic projection becomes evident, and we get for the admissible values of an electric charge $q = \pm 1$, i.e., a quantization rule!

In Sec. 6 we demonstrate the equivalence of the theory to geometrodynamics in a complexified Weyl-Cartan space. A reduction to Minkowski space identifies the magnetic monopole field as that of torsion and the Coloumb electric one as the ST Weyl nonmetricity. We conclude in Sec. 7 by the establishment of complex-valued Yang-Mills equations as the integrability conditions of GRCE and a discussion of general consequences of a complex-quaternionic structure of physical space. Finally, we discuss the relation of the AD approach to binary geometrophysics.

2. B-algebra and B-differentiability

Let $\mathbf{z}\in \mathrm{M}(4,\mathrm{C})$, $\mathbf{z} = \{z^\mu,\ \mu = 0,1,2,3\}$ be an element of the complex vector space M (4, C) of dimension $d = 4$. The function

$$\mathbf{F}(\mathbf{z}) = \{F^\mu(z)\} = \{F^\mu(z^0, z^1, z^2, z^3)\} \tag{1}$$

$\mathbf{F}\in\mathrm{M}$, maps an open domain $\mathrm{O}\subset\mathrm{M}$ to the domain $\mathrm{O}'\subset\mathrm{M}$; let its components $F^\mu(z)$ be complex and analytic.

Then a structure B of associative algebra of complex quaternions (biquaternions) $\mathrm{M}\times\mathrm{M}\to\mathrm{M}$ may be introduced on M . According to the isomorphism B = L(2, C), L being the full 2×2 complex matrix algebra, we shall use the matrix representation of B

$$\forall \mathbf{z} \in \mathrm{M} : \mathbf{z} = z^\mu \sigma_\mu = \begin{Vmatrix} u & w \\ p & v \end{Vmatrix}, \tag{2}$$

$\sigma_\mu = \{e, \sigma_a\}$, e being the unit 2×2 matrix and $\{\sigma_a,\ a = 1,2,3\}$ the Pauli matrices; $u, v = z^0 \pm z^3$; $p, w = z^1 \pm i z^2$ are the DeWitt coordinates on M . Now the multiplication $(*)$ in B is equivalent to the usual matrix one; the function (1) becomes a matrix-valued, or B-valued function of a B-variable. Let for some $\mathbf{z}\in\mathrm{O}$

$$d\mathbf{F} = \mathbf{F}(\mathbf{z} + d\mathbf{z}) - \mathbf{F}(\mathbf{z}) \tag{3}$$

be an infinitesimal increment (differential) of $\mathbf{F}(\mathbf{z})$, corresponding to a differential of a B-variable $d\mathbf{z}$ and according to the usual Euclidean metric $\rho^2 = \sum_\mu |z^\mu|^2$. Then we come to the following definition.

The function (1) $\mathbf{F}(\mathbf{z})$ is said to be B-*differentiable* in some domain $\mathrm{O}\subset\mathrm{M}$ if for $\forall \mathbf{z}\in\mathrm{O}$ there are some $\mathbf{G}(\mathbf{z})$, $\mathbf{H}(\mathbf{z})$ such that **the differential (3) may be presented in the invariant form**

$$d\mathbf{F} = \mathbf{G}(\mathbf{z}) * d\mathbf{z} * \mathbf{H}(\mathbf{z}), \tag{4}$$

i.e. only through the operation of multiplication in B .

For the commutative algebra of complex numbers, from (4) the Cauchy-Riemann (CR) equations follow in the coordinate representation, $\mathbf{F}' = \mathbf{G} * \mathbf{H}$ being a derivative of $\mathbf{F}(\mathbf{z})$. So the relation (4) naturally generalizes the CR equations to the case of a non-commutative associative B-algebra. Eqs. (4) will be further designated as GCRE (in the invariant form).

A detailed study of B-differentiability and analyticity, based on GCRE (4), may be found in [3], and a review of other approaches in [9]. The most profound is perhaps Fueter's work [10]; Gürsey et al. [11] applied it within the $d = 4$ gauge and chiral theories (see also [12]).

3. Spinor splitting and the eikonal equation

Let us turn now to the construction of field theory, based on the concept of B-differentiability. Consider a subspace $M_+ \subset M$ of the points with real coordinates $\mathbf{x} = \{x^\mu\} = \{z^\mu : \mathrm{Im}\,(z^\mu) = 0\}$, or else the subspace of Hermitian matrices with elements $\mathbf{z}^+ = \mathbf{z}$. The B-norm $N^2(\mathbf{z}) = \mathrm{Det}\,(\mathbf{z})$ then generates on M_+ the real Minkowski metric

$$N^2(\mathbf{x}) = \mathrm{Det}\,(\mathbf{x}) = uv - pw = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \tag{5}$$

so M_+ may be identified with the physical ST. Solutions to (4) on M may be obtained by analytic continuation from M_+. We will return to a detailed study of the relation between M and M_+ in Sec. 7.

It is now evident that the B-differentiable functions $\mathbf{F}(x)$, realizing the mappings $\mathbf{F}: M_+ \to M$, should be considered as a fundamental physical field; its spinor nature will be seen below. We will assume **the dynamics of a basic F-field to be completely determined by the GCRE (4) with $\mathbf{z} = \mathbf{x} \in M_+$**, i.e.

$$d\mathbf{F} = \mathbf{G}(x) * d\mathbf{x} * \mathbf{H}(x) \tag{6}$$

Except direct physical identifications of the abstract variables, in what follows no other assumptions will be necessary.

Let us rewrite now the matrices $\mathbf{F}$, $\mathbf{H}$ in (6) in the form

$$\mathbf{F} = \|\psi(x), \eta(x)\|, \quad \mathbf{H} = \|\alpha(x), \gamma(x)\| \tag{7}$$

each of ψ, η and α, γ being a matrix-column with two components; the columns transform independently through left multiplication. Then (6) splits into a pair of equations:

$$d\psi = \mathbf{G} * d\mathbf{x} * \alpha, \quad d\eta = \mathbf{G} * d\mathbf{x} * \gamma. \tag{8}$$

From (7) and (8) it follows that each solution to (6) may be presented in the form $\mathbf{F}(x) = \|\psi'(x), \psi''(x)\|$ where ψ', ψ'' are two arbitrary solutions of the unique irreducible equation

$$d\psi = \mathbf{G}(x) * d\mathbf{x} * \alpha(x). \tag{9}$$

The functions $\psi(x), \alpha(x)$ belong to the *left-side ideal* of the Clifford algebra $B = C(3,0)$ and are therefore obviously *2-spinors*. A conjugated spinor reduction of (6) is also possible, if the row splitting of $\mathbf{G}(x)$ is used; a double reduction may be realized as well.

These properties stand side by side with the widest symmetry group of Eqs. (4) or (6), including the transformations

$$\left.\begin{array}{ll} \mathbf{z} \to \mathbf{m} * \mathbf{z} * \mathbf{n}^{-1}, & \mathbf{F} \to \mathbf{k} * \mathbf{F} * \mathbf{l} \\ \mathbf{G} \to \mathbf{k} * \mathbf{G} * \mathbf{m}^{-1}, & \mathbf{H} \to \mathbf{n} * \mathbf{H} * \mathbf{l} \end{array}\right\}, \tag{10}$$

$\mathbf{m}, \mathbf{n}, \mathbf{k}, \mathbf{l}$ being arbitrary constant biquaternions of unit norm (neglecting the dilatations $\mathbf{z} \to \lambda\mathbf{z}$, $\lambda \in C$), $\mathbf{m}^{-1}, \mathbf{n}^{-1}$ are the inverse ones.

$\mathbf{Z}$-transformations in (10) define a 6C-parameter group of rotations $SO(4, C)$; the restriction of this group to M_+ (with $\mathbf{n}^{-1} = \mathbf{m}^+$) leads to the Lorentz transformations for $\mathbf{x}$. Now, if we put in (10) $\mathbf{k} = \mathbf{n}$, $\mathbf{l} = \mathbf{m}^{-1}$, the functions $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$ manifest their nature as 4-vectors ($\mathbf{F} \to \mathbf{n} * \mathbf{F} * \mathbf{m}^{-1}$ etc.). However, when $\mathbf{k} = \mathbf{n}$, $\mathbf{l} = \mathrm{Ident.}$, $\mathbf{G}$ transforms as a 4-vector, for $\mathbf{F}$ and $\mathbf{H}$ we have $\mathbf{F} \to \mathbf{n} * \mathbf{F}$, $\mathbf{H} \to \mathbf{n} * \mathbf{H}$, preserving the structure of the spinor splitting (8). Moreover, a double row-column splitting of (6) corresponds to the case $\mathbf{k} = \mathbf{l} = \mathrm{Ident.}$, when $\mathbf{F}(x)$ has to be considered as a scalar, while $\mathbf{G}$ and $\mathbf{H}$ transform as conjugated spinors $\mathbf{G} \to \mathbf{G} * \mathbf{m}^{-1}$, $\mathbf{H} \to \mathbf{n} * \mathbf{H}$.

Let us return now to the dynamical consequences of GCRE (6) and (9). Using the *Fiertz identity* and the double row-column splitting of (8), for every matrix component $\psi = \mathbf{F}_{AB}$; $A, B = 1, 2$ of $\mathbf{F}$-field we get [13]

$$(\partial_0\psi)^2 - (\partial_1\psi)^2 - (\partial_2\psi)^2 - (\partial_3\psi)^2 = 0. \tag{11}$$

Hence, **every matrix component of a B-differentiable function satisfies the nonlinear, Lorentz invariant, complexified eikonal equation** (11). For the B-algebra it plays a role similar to that of the Laplace equation in complex analysis; as for physics, its fundamental properties (for $\psi \in R$) were emphasized by V.A. Fock [14].

4. B-electrodynamics. Self-duality conditions and Maxwell equations

We shall further restrict ourselves to the case of the spinor equality $\alpha(x) = \psi(x)$ in (9), i.e. to the fundamental equation

$$d\psi = \mathbf{G}(x) * d\mathbf{x} * \psi(x). \tag{12}$$

For (12) the global continuous symmetries (10) are reduced to the transformations of the Lorentz group

$$\mathbf{x} \to \mathbf{m} * \mathbf{x} * \mathbf{m}^+, \ \psi \to \mathbf{s}\psi, \ \overline{\mathbf{G}} \to \mathbf{m} * \overline{\mathbf{G}} * \mathbf{m}^+, \tag{13}$$

where $\mathbf{s} = (\mathbf{m}^+)^{-1}$, $\overline{\mathbf{G}}$ is a B-conjugated field: $\overline{\mathbf{G}} * \mathbf{G} = (\mathrm{Det}\,\mathbf{G})^2$.

So relativistic invariance is ensured, and the conjugated field $\overline{\mathbf{G}}(x)$ forms a 4-vector. Later on $\overline{\mathbf{G}}(x)$ will be regarded as a C-valued matrix of electromagnetic (EM-) 4-potential $\mathbf{A}(x)$. Precisely, we set

$$A_\mu(x) = 2\overline{G}_\mu(x) \equiv 2G^\mu(x). \tag{14}$$

Such an identification will be justified further by its dynamic and geometric consequences, as well as by the establishment of gauge invariance of (12). Therefore, the latter **can be considered as the basic equations of B-electrodynamics**, i.e., some type of classical *spinor electrodynamics*, generated by solely the GCRE-structure.

Written in components, Eqs. (12) form the set of differential equations

$$\left.\begin{array}{l}\partial_u f = G^u f,\ \partial_p f = G^w f,\ \partial_w f = G^u h,\ \partial_v f = G^w h\\ \partial_u h = G^p f,\ \partial_p h = G^v f,\ \partial_w h = G^p h,\ \partial_v h = G^v h\end{array}\right\}. \quad (15)$$

Here $f(x)$ and $h(x)$ are the components of a 2-spinor field $\psi(x)$, and ∂ denotes a partial derivative with respect to the corresponding DeWitt coordinate.

The equations for the EM field follow from the overdetermined system (15) as its *integrability (compatibility) conditions*

$$\partial_\mu(\partial_\nu\psi) - \partial_\nu(\partial_\mu\psi) = 0, \quad \psi = \{f(x), h(x)\}.$$

Assuming then both $f(x)$, $h(x) \not\equiv 0$ (otherwise we would have obtained the same final results), we obtain after derivation

$$\left.\begin{array}{l}\partial_u A_w - \partial_p A_u = 0,\ \partial_w A_w - \partial_v A_u = \frac{1}{2}\,\mathrm{Det}\,\mathbf{A}\\ \partial_v A_p - \partial_w A_v = 0,\ \ \partial_p A_p - \partial_u A_v = \frac{1}{2}\,\mathrm{Det}\,\mathbf{A}\end{array}\right\}, \quad (16)$$

$A_\mu(x)$ being the EM potentials (14) and $\mathrm{Det}\,\mathbf{A} = A_u A_v - A_p A_w$. Going back to the Cartesian coordinates, we observe that Eqs. (16) are equivalent to the *self-duality conditions* (SDC)

$$\vec{\mathcal{P}} \equiv \vec{\mathcal{E}} + i\vec{\mathcal{B}} = 0 \quad (17)$$

for the C-valued electric $\vec{\mathcal{E}} = \{\mathcal{E}_a\}$ and magnetic $\vec{\mathcal{B}} = \{\mathcal{B}_a\}$ components of the EM field strength tensor

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu; \quad (18)$$

here

$$\mathcal{E}_a = \mathcal{F}_{0a},\ \mathcal{B}_a = \tfrac{1}{2}\,\varepsilon_{abc}\mathcal{F}_{bc};\ a,b,c,\ldots = 1,2,3. \quad (19)$$

In addition to (17), from (16) we have

$$\mathcal{D} \equiv \partial_\mu A^\mu + 2A_\mu A^\mu = 0, \quad (20)$$

i.e. an *inhomogeneous Lorentz condition.*

Combined with the definitions (18) and (19), the **SDC (17) lead then to the Maxwell equations in free space**

$$\partial_\nu \mathcal{F}^{\mu\nu} = \partial_\nu\left(\tfrac{1}{2i}\varepsilon^{\mu\nu\rho\lambda}\mathcal{F}_{\rho\lambda}\right) = 0. \quad (21)$$

So the Maxwell equations represent nothing but the *consistency conditions* of a basic GCRE-system and **are satisfied identically for each solution to the latter**. The inverse statement generally does not take place!

Now, it is easy to see that, according to the SDC (17), the *energy-momentum density* of a complex-valued EM-field turns to zero. Therefore, we ought to define the **physical** fields $\vec{E}$, $\vec{B}$ through the real (Re) or imaginary (Im) parts of (19). For geometric reasons (see part 6), we prefer

$$\vec{E} = 2\mathrm{Re}\,(\vec{\mathcal{E}}), \quad \vec{B} = 2\mathrm{Re}\,(\vec{\mathcal{B}}). \quad (22)$$

The R-valued vectors $\vec{E}$, $\vec{B}$ satisfy the linear Maxwell equations as well. However, they are mutually independent (contrary to (19)) and create a non-zero energy-momentum density (W, $\vec{P}$) of the usual form

$$W \sim \left(|\vec{E}|^2 + |\vec{B}|^2\right), \quad \vec{P} \sim [\vec{E}\times\vec{B}]. \quad (23)$$

Moreover, an infinite series of conservation laws can be obtained for (15) using routine procedures (see [15] for an example).

5. Coulomb field as a stereographic map. Electric charge quantization

Let us search now for solutions to the B-electrodynamic equations (15). Each of the two components $f(x)$, $h(x)$ of the spinor field in (15) satisfies the C-eikonal equation (11). Starting from one of its solutions, all the other quantities, including the EM potentials (14), should be derived. In particular, the wave-like solutions of (11) lead to EM fields, identical to the usual EM waves [3, 13].

Notice now that **the eikonal equation (11) possesses a wonderful invariance property** under the transformations

$$\psi(x) \to \Phi\left(\psi(x)\right) \quad (24)$$

with an arbitrary (C-differentiable) function $\Phi(f)$. Accordingly, one can easily verify the *gauge invariance* of the basic system (12) (and, therefore, (15)) of a special type:

$$\psi(x) \to \psi(x)\alpha(\psi),\ A_\mu(x) \to A_\mu(x) + \partial_\mu \ln\alpha(\psi), \quad (25)$$

$\alpha(\psi)$ being an arbitrary scalar function of the ψ components $f(x)$, $h(x)$.

The C-structure of the eikonal equation (11) essentially enlarges the spectrum of its solutions. The most important are certainly two static solutions found in [3]:

$$\left.\begin{array}{l} f^+ = \dfrac{x^1 + ix^2}{r + x^3} = \tan\left(\frac{\theta}{2}\right)\exp\left(i\varphi\right)\\ f^- = \dfrac{x^1 - ix^2}{r - x^3} = \cot\left(\frac{\theta}{2}\right)\exp\left(-i\varphi\right)\end{array}\right\}, \quad (26)$$

where $\{r, \theta, \varphi\}$ are usual spherical coordinates on R^3. From a geometric point of view, the expression (26) **corresponds to the stereographic projection** $\mathrm{S}^2 \to \mathrm{C}$ of a unit 2-sphere onto the C-plane (from the south (+) or north (-) poles, respectively). Substituting (26) into (15), (14), we get after trivial integration:

$$\left.\begin{array}{ll} f(x) = f^\pm(\theta,\varphi), & h(x) = \left[f^\pm(\theta,\varphi)\right]^2\\ A_u = \mp\dfrac{1}{r}, & A_v = \pm\dfrac{2}{r}\\ A_p = \dfrac{e^{-i\varphi}\left(\tan\frac{\theta}{2}\right)^{\mp 1}}{r}, & A_w = -\dfrac{2e^{i\varphi}\left(\tan\frac{\theta}{2}\right)^{\pm 1}}{r}\end{array}\right\} \quad (27)$$

or, for spherical components of the C-valued 4-potential

$$A_0 = \pm\frac{1}{2r}, \quad A_r = -\frac{1}{2r}, \quad A_\theta = \mp iA_\varphi = -\frac{1}{2r}\cot\theta \pm \frac{3}{2r\sin\theta}. \qquad (28)$$

Now, a transition to the physical vectors of EM-field strengths (22) shows that the *magnetic monopole* and gradient-like terms in (28) disappear and we get

$$E_\theta = E_\varphi = B_r = B_\theta = B_\varphi = 0, \quad E_r = \pm\frac{1}{r}, \qquad (29)$$

i.e. **the Coulomb law with a fixed value of electric charge** $q = \pm 1$.

Whereas the stereographic projection (26) and the transformations (24) realize the conformal mappings $S^2 \to C$, $C \to C$ respectively, EM fields behave by (25) in a gauge invariant manner, and the electric charge remains quantized. An exceptional role of conformal mappings in algebrodynamics has been clarified in [3] (chapter 1).

q-quantization is a crucial point for B-electrodynamics; a fundamental significance of this problem has been evident to Dirac, Eddington, Wheeler and other grands. In orthodox field theory the q-quantization is postulated rather than explained. The most elegant approach to this problem is produced, perhaps, by multidimensional ST theories [16]; B-electrodynamics presents another possibility.

In our approach the algebraic and purely classical origin of q-quantization becomes evident. The fact is that the initial GCRE are not invariant under the scaling $\mathbf{A} \to \lambda\mathbf{A}$, contrary to the linear Maxwell equations. We suppose, however, that the phenomenon of **"algebraic q-quantization"** should have deeper topological reasons; we hope to discuss them in the future [20].

6. Spinor connection and Weyl-Cartan geometry of ST

The fundamental equation of B-electrodynamics (12) may be presented in the form

$$\partial_\nu \psi = \mathbf{\Gamma}_\nu(x)\,\psi(x), \qquad (30)$$

with

$$\mathbf{\Gamma}_\nu(x) = \mathbf{G}(x) * \sigma_\nu \qquad (31)$$

being a *2-spinor connection* of special type. The initial GCRE, corresponding to (30), have the matrix form (6) with $\mathbf{H}(x) = \mathbf{F}(x)$, i.e.

$$d\mathbf{F} = \mathbf{G}(x) * d\mathbf{x} * \mathbf{F}(x), \qquad (32)$$

or, in a 4-vector representation [3]

$$\partial_\nu F^\mu = \Gamma^\mu_{\nu\rho}(x)\,F^\rho, \qquad (33)$$

$$\Gamma^\mu_{\nu\rho}(x) = 2(A_\nu\delta^\mu_\rho + A_\rho\delta^\mu_\nu - A^\mu\eta_{\nu\rho} - i\varepsilon^\mu_{\cdot\nu\rho\alpha}A^\alpha) \qquad (34)$$

where δ^μ_ν, $\eta_{\mu\nu}$, $\varepsilon_{\mu\nu\rho\lambda}$ are the Kronecker, Minkowski and Levi-Civita tensors, respectively, and $A_\mu(x)$ are C-valued potentials (14).

Thus in the basic electrodynamic case the initial GCRE system (6) is equivalent to the defining equations (32) of the *covariantly constant vector fields* $\{F^\mu(x)\}$ on a B-manifold with a "dynamically created" effective **geometry of Weyl-Cartan type**, represented by the affine connection (34). Note that the C-vector $A_\mu(x)$ completely determines both the Weyl part of (34) and its torsion structure. A generalization by introduction of a Riemann metric structure is natural as well.

To obtain the ST geometry induced by (34), let us pass from $\mathbf{F}(x)$ to the unitary field $\mathbf{U}(x) = \mathbf{F} * \mathbf{F}^+$. Using (32), we get

$$\partial_\nu U^\mu = \Delta^\mu_{\nu\rho}(x)\,U^\rho(x), \qquad (35)$$

with the R-valued connection

$$\Delta^\mu_{\nu\rho}(x) = 2(a_\nu\delta^\mu_\rho + a_\rho\delta^\mu_\nu - a^\mu\eta_{\nu\rho} - \varepsilon^\mu_{\cdot\nu\rho\alpha}b^\alpha), \qquad (36)$$

where $a_\mu(x)$ and $b_\mu(x)$ are real and imaginary parts of the potentials $A_\mu(x)$.

A connection similar to (36) has been introduced in Ref. [17] from physical considerations; in [18] it was shown to be **the only ST connection compatible with a spinor bundle structure** with the conventional notion of a covariant spinor derivative. In our approach these results follow from the GCRE structure alone.

However, the *torsion field* $b_\mu(x)$ in (36) satisfies the Maxwell equations, as well as the *non-metricity field* $a_\mu(x)$. By the key Ansatz (28), precisely the Weyl part $a_\mu(x)$ corresponds to the ordinary Coloumb electric field, justifying the previous identification of the EM field with the real part of the C-field.

As for the imaginary part $b_\mu(x)$, for (28) it has the magnetic monopole form

$$b_0 = b_r = b_\theta = 0, \quad b_\varphi = \mp\frac{1}{2r}\cot\theta - \frac{3}{2r\sin\theta}; \qquad (37)$$

we thus come to an exotic **geometric interpretation of magnetic monopoles as a ST torsion** (with a totally antisymmetric tensor structure). Accordingly, the field $b_\mu(x)$ cannot appear in the equations of geodesics. If we assume the latter to present the laws of test particle motion, then **monopoles should** have no effect on it and therefore **be entirely unobservable**!

Let us now return to the study of the primary C-geometry of the B-space. The integrability conditions for the irreducible spinor equation (30) may be written in the form

$$\mathbf{R}_{\mu\nu}\psi(x) = 0, \qquad (38)$$

with

$$\mathbf{R}_{\mu\nu} = \partial_{[\mu}\mathbf{\Gamma}_{\nu]} - [\mathbf{\Gamma}_\mu, \mathbf{\Gamma}_\nu] \qquad (39)$$

being the *curvature tensor* in the matrix representation. For its self-dual components

$$(\vec{\mathbf{R}})_a = \mathbf{R}_{0a} + \tfrac{i}{2}\varepsilon_{abc}\mathbf{R}_{bc} \tag{40}$$

with the connection of the form (31), we get

$$\vec{\mathbf{R}} = \vec{\mathcal{P}} + \mathcal{D}\vec{\sigma} - i[\vec{\mathcal{P}} \times \vec{\sigma}]. \tag{41}$$

Here the quantities

$$\vec{\mathcal{P}} = \vec{\mathcal{E}} + i\vec{\mathcal{B}}, \quad \mathcal{D} = \partial_\mu A^\mu + 2A_\mu A^\mu$$

coincide with (17) and (20), respectively and therefore vanish along with the entire self-dual tensor (40).

It is easy to see that $\mathcal{D}$ is proportional to the curvature invariant $\mathcal{D} = 6\eta^{\mu\nu}R^\alpha_{\mu\alpha\nu}$ $(= 0)$. So the **B-space appears to be an self-dual space with zero scalar curvature**. Now, if there are two **linearly independent** spinor solutions of (30), then, as follows from (38),

$$\mathbf{R}_{\mu\nu}(x) = 0, \tag{42}$$

i.e. a trivial case of flat geometry and zero field strengths.

To avoid that, the primary B-field $\mathbf{F}(x)$ in (32) should split into two spinors $\psi'(x)$, $\psi''(x)$ (see (9)), proportional to each other; therefore, we have

$$\operatorname{Det}\mathbf{F}(x) = 0, \tag{43}$$

and the field $\mathbf{F}(x)$ takes the values on the subspace of null divisors of the B-algebra, or, physically, **on the complex "light cone"**.

The null B-fields are the most fundamental objects throughout the AD approach as a whole. However, they can exist only on manifolds with an indefinite metric signature. So the **pseudo-Euclidean structure** of the World should not be postulated within the AD approach, but **is just a necessary condition of nontrivial dynamics (and effective geometry)**.

7. Yang-Mills fields and the C-structure of space-time

Now we will demonstrate that the Yang-Mills (YM) gauge fields also appear in theory in rather a natural way. To see that, let us separate the *trace-free part* in the basic spinor connection (31)

$$\mathbf{\Gamma}_\nu(x) = \mathbf{G}(x) * \sigma_\nu = \tfrac{1}{2}\big(A_\nu(x) + \mathbf{N}_\nu(x)\big). \tag{44}$$

Then the zero component $A_\mu(x)$ coincides with the C-potentials (14) of the EM field, and the trace-free part $\mathbf{N}_\mu(x)$ can be expressed in its terms in a linear way:

$$\mathbf{N}_\mu(x) = N^a_\mu(x)\,\sigma_a; \quad N^a_0 = A_a(x),$$
$$N^a_b = \delta_{ab}A_0(x) - i\varepsilon_{abc}A_c(x). \tag{45}$$

The quantities $\mathbf{N}_\mu(x)$ can be regarded as the matrix potentials of some C-valued gauge field; its strength corresponds to the traceless part of the curvature tensor (39) and may be written as usual:

$$\mathbf{L}_{\mu\nu} = \mathcal{L}^a_{\mu\nu}(x)\,\sigma_a = \partial_{[\mu}\mathbf{N}_{\nu]} - [\mathbf{N}_\mu, \mathbf{N}_\nu]. \tag{46}$$

We see now that the self-dual part of (46) coincides with the traceless part of the tensor (41) and, in view of (17) and (20), we have again

$$\mathbf{L}_{\mu\nu} + \tfrac{i}{2}\varepsilon_{\mu\nu\rho\lambda}\mathbf{L}^{\rho\lambda} = 0. \tag{47}$$

From (47) and the *Bianchi identity*, the YM equations follow immediately in a usual way:

$$\partial_\nu \mathbf{L}^{\mu\nu} = [\mathbf{N}_\nu, \mathbf{L}^{\mu\nu}]. \tag{48}$$

So we can indeed consider the field $\mathbf{N}_\nu(x)$ as a C-valued YM field of a special structure (45). From (38) for any non-trivial $\psi(x)$ we obtain, in addition,

$$\operatorname{Det}\mathbf{R}_{\mu\nu} = 0; \tag{49}$$

written in components, this condition leads to an expression of the EM field strength in terms of the YM ones (for each $[\mu\nu]$ separately):

$$\mathcal{L}^a_{\mu\nu}\mathcal{L}^a_{\mu\nu} = (\mathcal{F}_{\mu\nu})^2. \tag{50}$$

So **the EM field may be regarded as a modulus of the YM triplet field in the isotopic complexified 3-space.**

Contrary to EM fields, the YM ones cannot be split into real and imaginary parts (due to the nonlinearity of the YM equations) and therefore are essentially C-valued. This seems quite natural in connection with the pseudo-Euclidean structure of the ST (the duality operator is known to have imaginary eigenvalues in the Lorentz signature). Since the self-duality conditions play a crucial role both in orthodox field theory and in AD, we come again to the **conjecture on a C-analytic structure of real physical ST**. This possibility, discussed repeatedly within the frames of GRT, the twistor and string programs, as well as within the *binary geometrophysics* approach [19], seems to be inevitable in AD in view of the non-existence of a R-valued STA with an Aut group isomorphic to the Lorentz one. Thus, we suppose that close connections between the field equations nonlinearity and the C structure of ST do exist, as well as its noncommutative quaternion structure (see Sections 1 and 3 for the latter).

Moreover, **we may think of the C structure as some natural way of ST dimension enlarging** (namely, doubling), just in the sense of Kaluza-Klein theories. As for physics, such an effect should be essential at high energies; asymptotically, in the linear approximation, the ST C structure should split into the Minkowski space observed plus a conjugated one. The same is done by the field C-structure: it exhibits a reduction to a linear R-valued EM-field (doubled through the SDC (17), too).

Generally, we assume the existence of a biquaternion (i.e. complex-quaternion) algebraic structure of ST and field manifolds consistent with each other. The **non-commutativity** of such a B-algebra **results in the nonlinearity of fundamental dynamics**: the GCRE (6), **as well as its P- and even T-noninvariance** (the connections similar to (35), (36) are efficiently employed by V.G. Krechet for a 5-geometrical description of electroweak interaction). Nevertheless, here the usual reversible dynamics of gauge fields has been obtained in Sections 4 and 7; this latter should be regarded as nothing more but some "trace" of a primary B-structure, responsible for interactions, the "time arrow" and the left-right preference on the Minkowski ST. We expect an extensive presentation of our views of these problems in [20], as well as numerous generalizations of the AD approach.

In conlusion, peculiar correlations between AD and binary geometrophysics (BG) [19] should be noted. Both of the approaches start from some abstract exceptional algebraic structures and deal with either basic relations (in BG), or special mappings (in AD). In both theories **zero determinant structures** (see (43), (49)) **are of particular importance**. Finally, the ideas of *multipoint geometries* [19, 21] originate from purely algebraic considerations and should find their place in AD as well. It seems plausible that other deep interrelations will be found out in future.

We see that the simplest AD model, based on the conditions of B-differentiality alone, naturally contains the geometric, spinor-gauge and discrete structures, capable of solving the charge quantization and monopole problems. Within this model, the Coulomb law gains an exotic geometrical meaning, and the C-eikonal equation becomes a fundamental equation of field dynamics. Related problems (in particular, the problem of motion law and many-sources distributions) are yet to be solved.

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